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Enhanced D-brane categories from string field theory

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ABSTRACT: We construct D-brane categories in **B**-type topological string theory as solutions to string field equations of motion. Using the formalism of superconnections, we show that these solutions form an enhanced triangulated category as defined by Bondal and Kapranov. This analysis is an elaboration on recent work of Lazaroiu. We also comment on the relation between string field theory and the derived category approach of Douglas, and Aspinwall and Lawrence. Non-holomorphic deformations make a somewhat unexpected appearance in this construction.

KEYWORDS: Topological Field Theories, D-branes, BRST Symmetry, Differential and Algebraic Geometry.

Contents

1. Introduction

D-brane dynamics has been the object of much recent attention. There are roughly two main current directions of research in this area. Many of the recent papers are concerned with a fundamental microscopic description of D-branes by means of tachyon dynamics in open string field theory [25]–[39]. On the other hand, considerable effort has been made in order to improve our understanding of low energy effective dynamics of D-branes in situations with $N = 1$ supersymmetry [40]–[70]. In particular, Douglas [1] has proposed a beautiful formal structure underlying D-branes in topologically twisted $N = (2, 2)$ superconformal field theories. According to his work, and also to the detailed analysis of Aspinwall and Lawrence [2], we have to revise our traditional understanding of supersymmetric (even) branes on Calabi-Yau manifolds. Very briefly, they showed that D-branes are properly thought of as objects in a special category associated to a Calabi-Yau space X — the bounded derived category of coherent sheaves $D^b(X)$. In down to earth terms, this amounts to including differential complexes of coherent sheaves among physical D-branes.

In remarkable parallel work, Lazaroiu [4, 5, 6] has developed a very general approach to D-branes in string field theory. Using unitarity constraints and very general string field considerations, he has shown that D-branes naturally form certain enlarged categories equipped with special structures (such as a differential graded structure.) An axiomatic approach to topological open-closed string theories has been discussed in [3, 4].

The purpose of the present work is to study in more detail the D-brane category in topological string field theory, and to investigate the relation with the derived category of [1, 2]. Some elements along these lines have been sketched in [5, 6]. We take a pragmatic approach, by constructing an extension of Witten's holomorphic Chern-Simons theory [11]. The crucial element in this approach is the $\mathbb{Z}$ grading of topological boundary states introduced in [1]. This allows us to effectively identify the lowest mode expansion of the string field as a (graded) superconnection [13]. It turns out that the cubic topological string field theory reduces to a Chern-Simons theory for superconnections, using arguments similar to [11]. Note that superconnections have appeared in a similar context in [14]. The solutions to the string field equations of motion are shown to define *twisted complexes* as introduced by Bondal and Kapranov [17]. Very briefly, these are collections $\{E_n\}$ of holomorphic bundles with "maps" q_{mn} between various (non consecutive) E_m, E_n satisfying Maurer-Cartan equation. The precise definition of such objects will be given in sections 3 and 4. The associated categorical structure has been constructed by Bondal and Kapranov in [17]. According to the proposal of [5, 6], one should construct a more general category satisfying a certain completion condition.[1] We can avoid performing such a construction by restricting to the particular class of solutions described above.

The final result is the topological open string theories in this class form an *enhanced triangulated category* [17]. A generalized D-brane, i.e. an object in this category is a *twisted complex*, as sketched above. Perhaps one of the most striking aspects of this analysis is that, although we start with an open string background defined by holomorphic vector bundles E_n, we soon find general solutions of string field theory based on non-holomorphic deformations of the E_n. Nevertheless these determine consistent topological open string theories, with a good fermionic symmetry. Such solutions could be described as holomorphic superconnections.

A legitimate question is if these solutions define new topological branes or they are just artifacts of the string field approach. More specifically, one would like to know what is the relation between these D-brane categories and the derived categories found in [1, 2]. This is an interesting question, but we can provide only a partial answer in section 4. We show that the derived category is equivalent to a full subcategory of the D-brane category, by a careful comparison with [2]. However, we are unable to settle the question if these two categories are equivalent in spite of the apparent differences. This would prove that the string field approach brings nothing new. We expect that solving this puzzle would involve an alternative formulation of the string field category, perhaps in pure algebraic terms, if such a formulation exists. This is likely to be related to the approach of Kontsevich [18] in the context of homological mirror symmetry (see also [19].)

2. The topological B model

This is standard material, so we will review only what is needed. Recall that the standard $N = (2,2)$ superconformal algebra is generated by a set $T(z), G^{\pm}(z)$, $J(z)$, of holomorphic currents, and a similar set $\widetilde{T}(\bar{z})$, $\widetilde{G}^{\pm}(\bar{z})$, $\widetilde{J}(\bar{z})$ of anti-holomorphic cur-

[1] I thank C. Lazaroiu for pointing this out. See also [7].

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rents. It is common practice to introduce a bosonic representation of the $\mathrm{U}(1)$ current

$$J(z) = \partial\phi(z)\,, \qquad \widetilde{J}(\bar{z}) = \bar{\partial}\widetilde{\phi}(\bar{z})\,. \tag{2.1}$$

If the theory is formulated on the half-plane, one can impose either **A**-type of **B**-type boundary conditions preserving $\mathcal{N}=2$ superconformal symmetry [22]. In this paper, we will be exclusively concerned with **B** boundary conditions

$$G(z)^{\pm} = \widetilde{G}^{\pm}(\bar{z})\,, \qquad J(z) = \widetilde{J}(\bar{z})\,. \tag{2.2}$$

Note that the second equation is equivalent to Neumann boundary conditions for the compact boson $\phi(z,\bar{z})$.

Now let us discuss topological twists [9], following closely [10]. The main point is to alter the energy momentum tensor

$$\begin{aligned} T(z) &\longrightarrow T(z)_{\mathrm{top}} = T(z) \pm \frac{1}{2}\partial J(z)\,, \\ \widetilde{T}(\bar{z}) &\longrightarrow \widetilde{T}(\bar{z})_{\mathrm{top}} = \widetilde{T}(\bar{z}) \pm \frac{1}{2}\bar{\partial}\widetilde{J}(\bar{z})\,. \end{aligned} \tag{2.3}$$

Although it looks as if we have many choices, only the relative sign between the holomorphic and the anti-holomorphic part is relevant. This yields two types of topological string models dubbed again type **A**, when the signs are opposite and type **B**, when the signs are the same. To fix conventions, we will always take the sign of the holomorphic twist to be plus. The main effect is a shift of the conformal weight of all operators in the theory

$$h \longrightarrow h_{\mathrm{top}} = h - \frac{1}{2}q\,, \tag{2.4}$$

where q is the $\mathrm{U}(1)$ charge. The supercharge $G^{+}_{-1/2}$ becomes a nilpotent BRST charge Q in the twisted theory, $Q^2=0$. Accordingly, the $\mathrm{U}(1)$ charge becomes ghost charge, and the $\mathrm{U}(1)$ current $J(z)$ is simply the ghost number operator.

Next we consider open-closed topologically twisted models on the half-plane. By inspecting (2.2) and (2.3) it is clear that a **B** twist is compatible only with **B** boundary conditions, and this will be the case considered in this paper. This theory has been analyzed in the context of non-linear sigma models with Calabi-Yau target space in [11]. Let us denote by X the Calabi-Yau manifold. The main result of [11] is that the open string sigma model can be consistently coupled to a holomorphic bundle E on X, and the cubic string field theory action reduces in this case to a holomorphic Chern-Simons gauge theory.

In order to facilitate the presentation, it may be helpful to recall some details of the analysis of [11]. Let $\Phi : \Sigma \to X$ denote the map from the string world-sheet to the target space X. Recall [9, 11] that the fermi fields of the topological **B** model are $\eta^{\bar{i}}, \theta^{\bar{i}}$ sections of $\Phi^*(T^{0,1}(X))$ and ρ^i which is a section of $T^*(\Sigma) \otimes \Phi^*(T^{1,0}(X))$. We also define $\theta_j = g_{j\bar{i}}\theta^{\bar{i}}$. The lagrangian of the **B** model is

$$L = t \int_\Sigma d^2z \left(g_{i\bar{j}} \partial_z \phi^i \partial_{\bar{z}} \phi^{\bar{j}} + i\eta^{\bar{i}} (D_z \rho^i_{\bar{z}} + D_{\bar{z}} \rho^i_z) g_{i\bar{i}} + \right.$$
$$\left. + i\theta_i (D_{\bar{z}} \rho^i_z - D_z \rho^i_{\bar{z}}) + R_{i\bar{i}j\bar{j}} \rho^i_z \rho^j_{\bar{z}} \eta^{\bar{i}} \theta_k g^{k\bar{j}} \right), \tag{2.5}$$

which is invariant under the following fermionic symmetry

$$\begin{aligned} \delta\phi^i &= 0\,, \\ \delta\phi^{\bar{i}} &= i\epsilon\eta^{\bar{i}}\,, \\ \delta\eta^{\bar{i}} &= \delta\theta_i = 0\,, \\ \delta\rho^i &= -\epsilon d\phi^i\,. \end{aligned} \tag{2.6}$$

This model can be coupled to a background gauge field A on X via the boundary coupling

$$L_{\text{bdry}} = \int_{\partial\Sigma} \Phi^*(A) - \eta^{\bar{i}} F_{\bar{i}j} \rho^j\,. \tag{2.7}$$

It has been shown in [11] that this coupling preserves the fermionic symmetry if and only if

$$F_{\bar{i}\bar{j}}(A) = 0\,. \tag{2.8}$$

This means that the operator

$$\bar{\partial}_A = \bar{\partial} + A^{0,1} \tag{2.9}$$

defines an integrable holomorphic structure on the gauge bundle E. In the following we will fix such a background holomorphic bundle E, and we will adopt the notation $\bar{\partial}_E$ for the covariant Dolbeault operator.

Using standard string field theory arguments [11], the physical states can be found by computing the cohomology of Q on the kernel of the hamiltonian L_0 derived from (2.5). This has been done in [11], for the case at hand with the result that in the $t \to \infty$ limit the eigenfunctions localize on the subspace of constant maps $\Phi : I \to X$. Making use of the canonical commutation relations for fermions, we can write the string field as a wave functional depending on the zero modes of ϕ^I and $\eta^{\bar{i}}$

$$\Psi(\phi^I, \eta^{\bar{i}}) = a^0(\phi^I) + \eta^{\bar{i}} a^1_{\bar{i}}(\phi^I) + \eta^{\bar{i}}\eta^{\bar{j}} a^2_{\bar{i}\bar{j}}(\phi^I) + \eta^{\bar{i}}\eta^{\bar{j}}\eta^{\bar{k}} a^3_{\bar{i}\bar{j}\bar{k}}(\phi^I) + \cdots\,. \tag{2.10}$$

The components of the string field can be identified with $(0,q)$ forms on X where the degree q is the ghost number. The BRST operator Q in the background A can be shown to correspond to the Dolbeault operator $\bar{\partial}_E$, hence the physical states correspond to cohomology classes in $H^{0,q}(E^* \otimes E)$.

Since the degree q coresponds to the ghost charge, the only ghost number one term in (2.10) is the linear piece $a^1 \in \Omega^{0,1}(X, E^* \otimes E)$. This is to be interpreted as a deformation of the operator $\bar{\partial}_E$. Then, one can show [11] that the cubic string field

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action reduces in this case to holomorphic Chern-Simons theory

$$S = \frac{1}{2}\int_X \Omega \wedge \mathrm{Tr}\left(a^1 \wedge \bar{\partial}_E a^1 + \frac{2}{3} a^1 \wedge a^1 \wedge a^1\right), \tag{2.11}$$

where Ω is a non-vanishing holomorphic three-form on X. The equation of motion derived from this action is

$$\bar{\partial}_E a^1 + a^1 \wedge a^1 = 0\,. \tag{2.12}$$

which means that the deformed operator $\bar{\partial}_A + a^1$ is again integrable, therefore it defines a new holomorphic structure on E. Moreover, the gauge transformations of string field theory reduce in the $t \to \infty$ to ordinary complex gauge transformations

$$a^1 \longrightarrow a^1 + \bar{\partial}_E \epsilon\,. \tag{2.13}$$

Two deformations related by a gauge transformation give rise to isomorhic complex structures. It follows that the target space action of **B** topological field theory is intimately related to deformation of holomorphic bundles [20, 21].

To conclude this section, let us reformulate the above considerations in categorical language [5, 6]. The differential graded category of off-shell open string states can be defined as follows [5, 6]. The objects consist of holomorphic vector bundles E over X. Given two objects E, F, we define

$$\mathrm{Hom}_{\mathcal{E}}(E,F) = \oplus_{q=0}^{3} \Omega^{0,q}(E^* \otimes F)\,. \tag{2.14}$$

Note that $\mathrm{Hom}_{\mathcal{E}}(E,F)$ has a natural structure of graded abelian group, with the grading given by the ghost charge q. We also define a differential

$$d_{\mathcal{E}} : \mathrm{Hom}^q_{\mathcal{E}}(E,F) \longrightarrow \mathrm{Hom}^{q+1}_{\mathcal{E}}(E,F)\,, \qquad d_{\mathcal{E}} = \bar{\partial}_{E^* \otimes F}\,, \tag{2.15}$$

making $\mathrm{Hom}_{\mathcal{E}}(E,F)$ a differential complex. The associated cohomology category $H(\mathcal{E})$ is defined as having the same objects, and morphisms given by

$$\mathrm{Hom}_{H(\mathcal{E})}(E,F) = H(\mathrm{Hom}^{\bullet}_{\mathcal{E}}(E,F))\,. \tag{2.16}$$

One can also truncate consistently to cohomology in zero degree in (2.16), obtaining another cohomology category $H^0(\mathcal{E})$. The physical intepretation should be clear: the objects represent topological D-branes, while the morphisms represent the off-shell open string states in the presence of two D-branes E and F. The differential is the BRST operator which defines physical states. Passing to the cohomology category is equivalent to keeping only physical open string states. We will see later that if we introduce a grading of boundary states, we will find a much bigger D-brane category.

3. Grading and a generalization of holomorphic Chern-Simons theory

In this section we present a generalization of the previous analysis which takes into account the $\mathbb{Z}$ grading of boundary states discovered in [1]. This is one of the main ingredients of [1] in establishing the relation between boundary states in topological models and derived categories. In order to explain the main idea, recall that the theory contains a U(1) current $J = \partial\phi$ which becomes ghost number operator in the topological model. For **B** models, the compact boson ϕ is subject to Newmann boundary conditions. This means that the open string states stretching between two D-branes E, F will carry a quantum number representing KK momentum around the circle. In the topological theory, this quantum number is a boundary ghost charge. For **A** models, the same quantum number has been described as winding number [1]. Since we are working in off-shell string field theory, the effect of this quantum number is to induce a grading on the space of boundary states. In other words, we have to distinguish between a D-brane E and a D-brane defined by an isomorphic holomorphic bundle if there is an open string with boundary ghost charge p stretching between them. More concretely, this means that a D-brane must be specified by a holomorphic Chan-Paton bundle on X together with an integer $n \in \mathbb{Z}$. The open string states stretching between the D-branes E_n and E_{n+p} will carry p units of ghost charge.

Given these considerations we can now proceed with the analysis of string field theory in the background of a graded collection of D-branes $\{E_n\}$. The E_n are holomorphic vector bundles over X. Note that this is not the most general configuration possible since one assumes that the bundle E_n has grade n. Hence no two of the countable family of bundles considered in the theory can have the same grade, which is a serious constraint in the D-brane configurations allowed.

First, we write down the most general expansion of the string field

$$\Psi(\phi^I, \eta^{\bar{i}}) = \sum_{m,n} a^0_{mn} + \eta^{\bar{i}}(a^1_{\bar{i}})_{mn} + \eta^{\bar{i}}\eta^{\bar{j}}(a^2_{\bar{i}\bar{j}})_{mn} + \eta^{\bar{i}}\eta^{\bar{j}}\eta^{\bar{k}}(a^3_{\bar{i}\bar{j}\bar{k}})_{mn}\,, \tag{3.1}$$

where c_{mn} are maps from E_m to E_n i.e. sections of $E_m^* \otimes E_n$. Similarly, the higher order terms can be regarded as sections of $\Omega^{0,q}(X) \otimes (E_m^* \otimes E_n)$. Keeping in mind the relation between grading and ghost number, it follows that a section of $\Omega^{0,q}(X) \otimes (E_m^* \otimes E_n)$, has ghost number $q + (n - m)$. Therefore the ghost number one piece of $\Psi(\phi^I, \eta^{\bar{i}})$ is

$$\Psi^{(1)}(\phi^I, \eta^{\bar{i}}) = \sum_{n} a^0_{n,n+1} + \eta^{\bar{i}}(a^1_{\bar{i}})_{n,n} + \eta^{\bar{i}}\eta^{\bar{j}}(a^2_{\bar{i}\bar{j}})_{n+1,n} + \eta^{\bar{i}}\eta^{\bar{j}}\eta^{\bar{k}}(a^3_{\bar{i}\bar{j}\bar{k}})_{n+2,n}\,. \tag{3.2}$$

The next step is to compute the cubic string field action in terms of the components of (3.2).

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At this stage it may be helpful to discuss some of the mathematical structure underlying open string field theory [12]. The string fields form an associative non-commutative graded algebra $\mathcal{A}$, the grading being defined by the ghost number. This algebra is endowed with a derivation Q, which is the BRST operator, and with a trace map $\int : \mathcal{A} \to \mathbb{C}$, satisfying the following rules

$$\begin{aligned} Q(a * b) &= (Qa) * b + (-1)^{\deg(a)} a * (Qb)\,, \\ \int a * b &= (-1)^{\deg(a)\deg(b)} \int b * a\,, \\ \int Qa &= 0\,. \end{aligned} \tag{3.3}$$

The structure in (3.3) defines a differential graded algebra.[2]

Following the strategy of [11], we first give a more concrete description of this algebra for topological open strings in the large t limit. In this case, the string field admits an expansion in terms of lowest lying modes which can be identified with elements of $\Omega^{0,q}(E_m^* \otimes E_n)$. Therefore the underlying space of the algebra would naively be identified with

$$\oplus_{q=0}^{3} \oplus_{(m,n)\in\mathbb{Z}^2} \Omega^{0,q}(E_m^* \otimes E_n)\,, \tag{3.4}$$

with the grading defined by the ghost number $q + (n - m)$. In fact, it turns out that working with $\mathbb{Z}$-graded bundles is not quite enough in order to reproduce the structure of the string field algebra $\mathcal{A}$. In order to reproduce the relations (3.3) the naive proposal (3.4) must be refined by working with $\mathbb{Z}$-graded *super* vector bundles. In other words, each graded vector bundle $\{E_n\}$ can be viewed as a graded super vector bundle $\{\check{E}_n\}$, where

$$\check{E}_n = \begin{cases} (E_n, 0) & \text{for } \; n \text{ even}\,, \\ (0, E_n) & \text{for } \; n \text{ odd}\,. \end{cases} \tag{3.5}$$

This is a standard construction [18]. Given such an object, we can obtain either a $\mathbb{Z}$-graded bundle by forgetting the $\mathbb{Z}/2$ grading or a super vector bundle by forgetting the $\mathbb{Z}$-grading. In the last case, the resulting $\mathbb{Z}/2$-graded bundle $\check{E} = (E^+, E^-)$ has components

$$E^+ = \oplus_k E_{2k}\,, \qquad E^- = \oplus_k E_{2k+1}\,. \tag{3.6}$$

Now, note that we have a superalgebra $\Omega(X) = \oplus_{q=0}^{3} \Omega^{0,q}(X)$ (with standard multiplication of forms), and another superalgebra $\mathrm{End}(\check{E}) = \oplus_{(m,n)\in\mathbb{Z}^2} \Omega^{0,0}(E_m^* \otimes E_n)$. In the second case, there is an extra $\mathbb{Z}$ grading defined by $(n - m)$; if we ignore this

[2]The axioms (3.3) do not suffice to describe a theory with D-branes. As discussed in [5], what is missing is a category structure. It is important to check that the string field product and metrics decompose in a manner consistent with this structure.

grading, the superalgebra structure is the standard one [13]. Now we can construct the $\mathbb{Z}$-graded superalgebra

$$\mathcal{A} = \Omega(X) \otimes_{\Omega^{0,0}(X)} \mathrm{End}(\check{E})\,. \tag{3.7}$$

In order to keep track of various gradings, we introduce the notation $\mathcal{A}^q_{(m,n)} = \Omega^{0,q}(E^*_m \otimes E_n)$, so that we have

$$\mathcal{A} = \oplus_{m,n,q}\mathcal{A}^q_{(m,n)}\,. \tag{3.8}$$

The degree of an element $f \in \mathcal{A}^q_{m,n}$ is $\deg(f) = q + (n-m)$. Again, if we ignore the $\mathbb{Z}$-grading, this is the standard tensor product of superalgebras [13]. If $\omega, \eta \in \Omega(X)$, and $f, g \in \mathrm{End}(\check{E})$, we have

$$(\omega \otimes f)(\eta \otimes g) = (-1)^{(n-m)\deg(\eta)}(\omega \wedge \eta)(fg)\,. \tag{3.9}$$

A similar construction can be found in a different context in [15].

We would like to claim that this is the correct construction of the topological open string algebra. This needs some further justification justification, as in [65, section 6].[3] The string field (3.1) is an extended string field [16], that is a superfield defined on a supermanifold with local Grassmann odd coordinates $\eta^{\bar{i}}$. Moreover, in the present conventions, Ψ is a Grassmann odd superfield, i.e. we have the commutation relations

$$a^q_{m,n}\eta^{\bar{i}} = (-1)^{1-q}\eta^{\bar{i}}a^q_{m,n}\,. \tag{3.10}$$

To each monomial of the form $\eta^{\bar{i}_1}\cdots\eta^{\bar{i}_q}a^q_{\bar{i}_1\cdots\bar{i}_q}$ we can associate a Grassmann valued form $a = d\bar{z}^{\bar{i}_1}\wedge\cdots\wedge d\bar{z}^{\bar{i}_q}$. These are elements of the above superalgebra and constitute components of an unextended string field. The problem is that multiplication of extended string fields induces a product

$$a * b = (-1)^{(1-q+m-n)p}ab\,, \tag{3.11}$$

where $a \in \mathcal{A}^q_{m,n}, b \in \mathcal{A}^p_{n,s}$. The product in the right-hand side is superalgebra multiplication. So we get an agreement only if $q + (n-m) = 1$, i.e. for states of total ghost number one. This is good enough for our purposes since we will be concerned only with ghost number one superfields form now on. Moreover, we will write all superfields as unextended superfields, without explicitly changing notation.

Let us describe the remaining elements. The trace map $\int : \mathcal{A} \to \mathbb{C}$ is given by

$$\int f = \int \Omega \wedge \mathrm{Tr}_s(f)\,, \tag{3.12}$$

where $\mathrm{Tr}_s : \mathcal{A} \to \Omega(X)$ denotes the supertrace of [13]. It is a standard fact that

$$\mathrm{Tr}_s(fg) = (-1)^{\deg(f)\deg(g)}\,\mathrm{Tr}_s(gf) \tag{3.13}$$

[3] I thank the reviewer for pointing this out.

for any two elements $f, g \in \mathcal{A}$. The BRST operator is a superconnection $D : \mathcal{A} \to \mathcal{A}$ satisfying the Leibniz rule [13]

$$D(\omega f) = \bar{\partial}\omega + (-1)^{\deg(\omega)} \omega D f \tag{3.14}$$

for all $\omega \in \Omega(X)$, $f \in \mathcal{A}$. In the particular case under study, D has the special form

$$D = \oplus_{n \in \mathbb{Z}} (\bar{\partial}_{E_n}) \, . \tag{3.15}$$

Given, (3.13) and (3.14), one can check that the relations (3.3) are satisfied.

We are now ready to write down the cubic action of topological open string field theory. Recall that we have to consider only the ghost number one piece which is reproduced below for convenience

$$\Psi^{(1)} = \sum_n a^0_{n,n+1} + a^1_{nn} + a^2_{n+1,n} + a^3_{n+2,n} \, , \tag{3.16}$$

where $a^q_{m,n} \in \mathcal{A}^q_{m,n}$. Using the graded superalgebra structure discussed so far, the cubic action can be written in compact form

$$S = \frac{1}{2} \int_X \Omega \wedge \mathrm{Tr}_s \left(\Psi^{(1)} D \Psi^{(1)} + \frac{2}{3} \Psi^{(1)} \Psi^{(1)} \Psi^{(1)} \right) . \tag{3.17}$$

This is the super extension of the holomorphic Chern-Simons action mentioned in the introduction. Substituting (3.16) into (3.17) we obtain the following expression

$$\begin{aligned} S = \frac{1}{2} \int_X \Omega \wedge \Bigg[& \mathrm{Tr}_s \left(a^1 (D a^1) + a^0 (D a^2) + a^2 (D a^0) \right) + \\ & + \frac{2}{3} \mathrm{Tr}_s \left(a^1 a^1 a^1 + a^0 a^0 a^3 + a^0 a^3 a^0 + a^3 a^0 a^0 \right) + \\ & + \frac{2}{3} \mathrm{Tr}_s \left(a^0 a^1 a^2 + (\text{all permutations}) \right) \Bigg] , \end{aligned} \tag{3.18}$$

where a^q is a sum over all n. For example

$$a^0 = \sum_n a^0_{n,n+1} \tag{3.19}$$

and so on. Reasoning by analogy with the ungraded case, we can view $\Psi^{(1)}$ as a deformation of the superconnection D defined in (3.15). Since the bundles E_n that we started with are holomorphic, D satisfies the integrability condition $D^2 = 0$. We will refer to this condition as flatness. The equations of motion derived from (3.17) read

$$D \Psi^{(1)} + \Psi^{(1)} \Psi^{(1)} = 0 \, , \tag{3.20}$$

or, in components,

$$\begin{aligned} a^0_{n+1,n+2} a^0_{n,n+1} &= 0 \, , \\ D a^0_{n,n+1} + a^0_{n,n+1} a^1_{n,n} + a^1_{n+1,n+1} a^0_{n,n+1} &= 0 \, , \\ D a^1_{n,n} + a^1_{n,n} a^1_{n,n} + a^2_{n+1,n} a^0_{n,n+1} + a^0_{n-1,n} a^2_{n,n-1} &= 0 \, , \\ D a^2_{n+1,n} + a^1_{n,n} a^2_{n+1,n} + a^2_{n+1,n} a^1_{n+1,n+1} + a^3_{n+2,n} a^0_{n+1,n+2} + a^0_{n-1,n} a^3_{n+1,n-1} &= 0 \, . \end{aligned} \tag{3.21}$$

An important point is that these equations are equivalent to flatness of the deformed superconnection $D + \Psi^{(1)}$. Therefore on-shell string field configurations are determined by flat superconnections of a general form (as opposed to D, which is a diagonal superconnection.) Formulated differently, the above argument shows that, given a topological open string theory defined by the collection of D-branes $\{E_n\}$, we can deform by arbitrary operators $a^0_{n,n+1}, a^1_{n,n}, \ldots$. One obtains consistent topological open string theories as long as the flatness conditions (3.20) and (3.21) are satisfied.

It is interesting to note that eqs. (3.21) do not enforce holomorphic deformations of the bundles E_n. Namely, consider the second equation in (3.21)

$$Da^1_{n,n} + a^1_{n,n}a^1_{n,n} + a^2_{n+1,n}a^0_{n,n+1} + a^0_{n-1,n}a^2_{n,n-1} = 0\,. \tag{3.22}$$

$a^1_{n,n}$ is a deformation of the covariant Dolbeault operator $\bar\partial_{E_n}$. As noted before, this deformation defines a new holomorphic structure if and only if

$$\bar\partial_{E_n} a^1_{n,n+1} + a^1_{nn}a^1_{nn} = 0\,. \tag{3.23}$$

Therefore eq. (3.22) allows non-holomorphic deformations of the E_n at the price of exciting the higher q-form fields a^q, $q \geq 2$. We do not know at this stage if these are genuine new deformations of the topological **B** model. For example, we can turn values of the higher fields a^q such that holomorphy is preserved. These would correspond to the on-shell deformations of [1, 2], where it has been shown that they do not give anything new beyond the derived category. The problematic deformations are the non-holomorphic ones. This can be hopefully settled by searching for examples in concrete models, and we leave this for future work.

In the following we try to elucidate the categorical structure of the D-branes found above and comment on the relation with derived categories.

4. Twisted complexes and enhanced triangulated categories

In this section we will show that the solutions to string field theory found above form an *enhanced triangulated category* as defined by Bondal and Kapranov [17]. Moreover, this category turns out to include the bounded derived category $D^b(X)$ as a full subcategory, therefore the result is consistent with previous work on the subject [1, 2]. We stress that we will not attempt to settle the question whether these two categories are equivalent. If they were equivalent, this would mean that the string field approach brings nothing new. In that case, it would still be interesting to have an explicit construction of the equivalence. This section follows ideas proposed in [6, section 5]. The relation with twisted complexes of Bondal and Kapranov, enhanced triangulated categories, as well as the derivation of (an extension of) $D^b(X)$ from string field theory have been already discussed there. Since their discussion is brief, we spell out some details below.

In order to avoid any technical complications, we will consider only Chan-Paton bundles of finite rank, as in [2]. Hence all but finitely many E_n will be zero, and we are dealing with the bounded derived category $D^b(X)$. Note though that there is no convincing reason for this restriction from what has been said so far. In fact, from the point of view of string field theory it appears to be more natural to work with infinite complexes (see also [23, 24].) We will not pursue this further in the present paper.

We start by a giving a more formal description of the set of solutions to the equations of motion (3.20) and (3.21). Recall that at the end of section 2, we have introduced a DG category $\mathcal{E}$ whose objects are holomorphic vector bundles on X. The morphisms are given by

$$\mathrm{Hom}_{\mathcal{E}}(E,F) = \oplus_{q=0}^{3}\Omega^{0,q}(E^*\otimes F)\,, \tag{4.1}$$

which is a graded abelian group with differential

$$d_{\mathcal{E}} = \bar{\partial}_{E^*\otimes F}\,. \tag{4.2}$$

The cohomology of the morphism complex $\mathrm{Hom}^{\bullet}_{\mathcal{E}}(E,F)$ describes the physical operators of topological open string theory in the presence of two D-branes E and F.

Bondal and Kapranov [17] introduced a formal construction which associates to any DG category an enlarged DG category whose objects are *twisted complexes*. This enhanced category has been denoted by Pre-Tr($\mathcal{E}$) in [17]. We show below that their twisted complexes are precisely in one to one correspondence to solutions to (3.21). A twisted complex is a collection $\{E_n\}$ of objects of $\mathcal{E}$ with morphisms $q_{mn}\in\mathrm{Hom}^{m-n+1}(E_m,E_n)$ satisfying the equation

$$d_{\mathcal{E}}q_{mn} + \sum_p q_{pn}q_{mp} = 0\,. \tag{4.3}$$

Given (4.1), it follows that the morphisms q_{mn} are forms in $\Omega^{m-n+1}(E^*_m\otimes E_n)$. By comparing with (3.2), it follows that the q_{mn} are precisely the components of string field of ghost number one in the presence of a graded collection of D-branes $\{E_n\}$. Moreover, eqs. (4.3) are equivalent to the equations of motion (3.21), therefore the twisted complexes (E_n,q_{mn}) describe on-shell string field configurations. This is certainly a promising start, as the construction of Bondal and Kapranov was not originally related to string theory in any way.

Now let us describe the morphisms of Pre-Tr($\mathcal{E}$), that is to each pair of objects $C=\{E_n,q_{mn}\}$, $C'=\{E'_n,q'_{mn}\}$, we associate a graded abelian group $\mathrm{Hom}_{\text{Pre-Tr}(\mathcal{E})}$ (C,C'), with a differential $d_{\text{Pre-Tr}(\mathcal{E})}$. We have [17]

$$\mathrm{Hom}^k_{\text{Pre-Tr}(\mathcal{E})}(C,C') = \oplus_{q+m-n=k}\,\mathrm{Hom}^q_{\mathcal{E}}(E_m,E'_n)\,, \tag{4.4}$$

and for $f_{mn} \in \mathrm{Hom}^q_{\mathcal{E}}(E_m, E'_n)$

$$d_{\text{Pre-Tr}(\mathcal{E})} f_{mn} = d_{\mathcal{E}} f_{mn} + \sum_p q'_{pn} f_{mn} + (-1)^{q(m-p+1)} f_{mn} q_{mp} \,. \tag{4.5}$$

This defines a DG structure on Pre-Tr($\mathcal{E}$). Although this construction looks rather complicated, let us note that it has a natural physical interpretation. We noticed before that the twisted complexes are nothing else than solutions to string field theory which define new deformed topological string theories. The deformations are encoded by the maps q_{mn}. Each deformation of the topological field theory should reflect in a deformation of the BRST operator, as in [1, 2]. The formula (4.5) defines the deformed BRST operator corresponding to two generalized D-branes defined by the objects C, C'. It acts on the graded space $\mathrm{Hom}^{\bullet}_{\text{Pre-Tr}(\mathcal{E})}(C, C')$ which represents the space of open string states stretching between C and C'. The grading is induced by ghost number.

In order to complete the picture, we need to take one last step, namely to keep only the physical open string states. This can be achieved by passing to the cohomology category associated to $\mathcal{E}$. This means that we keep the same objects, but we replace the morphisms $\mathrm{Hom}^{\bullet}_{\text{Pre-Tr}(\mathcal{E})}(C, C')$ by the graded abelian space $H(\mathrm{Hom}^{\bullet}_{\text{Pre-Tr}(\mathcal{E})}(C, C'))$. In other words, we take BRST cohomology on the open string states, keeping only inequivalent physical states in each ghost degree. We will denote the resulting category by $H(\text{Pre-Tr}(\mathcal{E}))$.

Strictly speaking, keeping all ghost degrees might be superfluous. It should suffice to restrict to cohomology of degree zero $H^0(\mathrm{Hom}_{\text{Pre-Tr}(\mathcal{E})}(C, C'))$, in which case we obtain the category Tr($\mathcal{E}$) defined in [17]. For example a similar phenomenon takes place for the class of topological open string theories considered in [1, 2]. That is, there is no loss of information if one keeps only the cohomology of degree zero of the D-brane category $\mathbf{T}(X)$ defined in [2]. The higher cohomology is recovered by applying the shift functor. So we conjecture that the category encoding all the physical information is $H^0(\text{Pre-Tr}(\mathcal{E})) = \text{Tr}(\mathcal{E})$. An encouraging fact is that Tr($\mathcal{E}$) is a triangulated category, which should be a key element in defining a suitable notion of stability [1]. However, a more detailed study would be desirable, and will be pursued elsewhere.

To conclude this section, let us investigate in some detail the relation between the enlarged D-brane category found in this section and the derived category $D^b(X)$. We will only be able to show that the derived category is equivalent to a full subcategory of Tr($\mathcal{E}$), using the model of [2] for $D^b(X)$. To this end, let us consider the full subcategory of Pre-Tr($\mathcal{E}$) generated by objects $C = \{E_n, q_{mn}\}$ which are complexes i.e. $q_{mn} = 0$, unless $n = m + 1$. In this case, the relations (4.3) reduce to

$$\begin{aligned} d_{\mathcal{E}} q_{m,m+1} &= 0 \,, \\ q_{m,m+1} q_{m-1,m} &= 0 \,, \end{aligned} \tag{4.6}$$

which show that C is a holomorphic complex of holomorphic vector bundles. Let us analyze the morphisms in $\mathrm{Tr}(\mathcal{E})$ between two such objects. For this, we have to specialize the formulae (4.4) and (4.5) to the case at hand, i.e. for two complexes C, C'

$$\begin{aligned} \mathrm{Hom}^k_{\text{Pre-Tr}(\mathcal{E})}(C, C') &= \oplus_{q=0}^{3} \oplus_m \Omega^{0,q}(E_m^* \otimes E'_{m+k-q}) \,, \\ d_{\text{Pre-Tr}(\mathcal{E})} f_{m,m+k-q} &= \bar\partial_{E_m^* \otimes E'_{m+k-q}} f_{m,m+k-q} + q'_{m+k-q+1,m+k-q} f_{m,m+k-q} + \\ &\quad + f_{m,m+k-q} q_{m,m+1} \,, \end{aligned} \tag{4.7}$$

where $f_{m,m+k-q} \in \Omega^{0,q}(E_m^* \otimes E'_{m+k-q})$. This yields a differential complex, whose cohomology defines $\mathrm{Hom}_{H(\text{Pre-Tr}(\mathcal{E}))}(C, C')$; the cohomology of degree zero defines $\mathrm{Hom}_{\mathrm{Tr}(\mathcal{E})}(C, C')$. Let $\mathbf{S}(X)$ denote the full subcategory of $H(\text{Pre-Tr}(\mathcal{E}))$ generated by complexes; $\mathbf{S}_0(X)$ will denote the corresponding full subcategory of $\mathrm{Tr}(\mathcal{E})$.

Our goal is to compare $\mathbf{S}(X)$, $\mathbf{S}_0(X)$ with $\mathbf{T}(X)$, $\mathbf{T}_0(X)$ defined in [2, section 2]. Recall that the objects of $\mathbf{T}(X)$ are holomorphic complexes of holomorphic vector bundles on X, therefore it has the same objects as $\mathbf{S}(X)$. The morphisms of $\mathbf{T}(X)$ are graded abelian groups defined as the cohomology of the double complex

$$\begin{array}{ccccc} & \bar\partial\big\uparrow & & \bar\partial\big\uparrow & \\ \xrightarrow{\bar q} & \Omega^{0,1}(\underline{\mathrm{Hom}}^0(E_\bullet, E'_\bullet)) & \xrightarrow{\bar q} & \Omega^{0,1}(\underline{\mathrm{Hom}}^1(E_\bullet, E'_\bullet)) & \xrightarrow{\bar q} \\ & \bar\partial\big\uparrow & & \bar\partial\big\uparrow & \\ \xrightarrow{\bar q} & \Omega^{0,0}(\underline{\mathrm{Hom}}^0(E_\bullet, E'_\bullet)) & \xrightarrow{\bar q} & \Omega^{0,0}(\underline{\mathrm{Hom}}^1(E_\bullet, E'_\bullet)) & \xrightarrow{\bar q} \\ & \bar\partial\big\uparrow & & \bar\partial\big\uparrow & \end{array} \tag{4.8}$$

where

$$\underline{\mathrm{Hom}}^q(E_\bullet, E'_\bullet) = \oplus_m \underline{\mathrm{Hom}}(E_m, E'_{m+q}) \,. \tag{4.9}$$

The horizontal differential $\bar q$ is given by

$$\bar q f_{mn} = q'_{n,n+1} f_{mn} + f_{mn} q_{m-1,m} \,, \tag{4.10}$$

which is in fact identical to the non-derivative part of the second equation in (4.7). It is now a straightforward exercise to check that the complex defined in (4.7) is the simple complex associated to the double complex (4.8). This shows that they have isomorphic cohomology, therefore the categories $\mathbf{S}(X)$ and $\mathbf{T}(X)$ are equivalent. The same is true for $\mathbf{S}_0(X)$ and $\mathbf{T}_0(X)$. On the other hand, one of the main results of [2] is that $\mathbf{T}_0(X)$ is equivalent to the derived category $D^b(X)$. Hence we have effectively identified $D^b(X)$ with a full subcategory of $\mathrm{Tr}(\mathcal{E})$.

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